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Upper bounds for turbulent Couette flow incorporating the poloidal power constraint

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The upper bound on momentum transport in the turbulent regime of plane Couette flow is considered. Busse (1970) obtained a bound from a variational formulation based on total energy conservation and the mean momentum equation. Two-dimensional asymptotic solutions of the resulting Euler–Lagrange equations for the system were obtained in the large-Reynolds-number limit. Here we make a toroidal poloidal decomposition of the flow and impose an additional power integral constraint, which cannot be satisfied by two-dimensional flows. Nevertheless, we show that the additional constraint can be met by only small modifications to Busse's solution, which leaves his momentum transport bound unaltered at lowest order. On the one hand, the result suggests that the addition of further integral constraints will not significantly improve bound estimates. On the other, our optimal solution, which possesses a weak spanwise roll in the outermost of Busse's nested boundary layers, appears to explain the three-dimensional structures observed in experiments. Only in the outermost boundary layer and in the main stream is the solution three-dimensional. Motion in the thinner layers remains two-dimensional characterized by streamwise rolls.

1. Introduction

Attempts to understand turbulent fluid flows based on suitably averaged equations invariably encounter the well-known 'closure' problem in which there are always more unknowns than equations. Progress is then dependent upon the introduction of a certain number of heuristic assumptions which inevitably compromise the final deductions. Upper-bound theory represents an alternative approach free from such assumptions in which rigorous bounds on chosen mean quantities are derived. The idea is to seek an extreme of some averaged flow quantity from a manifold of vector fields which satisfy only a reduced number of constraints implied by the complete equations describing the flow. Since the realized solutions are contained within this manifold, the deduced extreme then acts to bound the observed values. In principle, this bound can be improved by incorporating further information from the governing equations in the form of additional constraints until ultimately the set of vector fields must be the solution set and the bound is attained. In practice, the variational problems, which arise, quickly approach the complexity of the full system as constraints are added. The hope then is that the upper bound and optimizing

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vector field may start to reflect features of the turbulent solution before this point is reached (see Howard 1972 and Busse 1978 for reviews).

Formal upper-bound theory has its origins in the work of Howard (1963) who considered turbulent convection in a layer heated from below. Following ideas by Malkus (1954), Howard formulated and solved a variational problem to maximize the convective heat transport subject only to the two dissipation-rate integrals of the basic Boussinesq equations. Imposing the continuity equation as an extra constraint led to a substantially more involved problem to which Howard sought a separable solution. As a generalization of this, Busse (1969*a*) then discovered a new class of non-separable solutions to this problem which increased the bound for sufficiently large Rayleigh numbers. Encouraged by the similarity of his 'multi-α' solution with experimental observations, Busse (1970, referred to below as B70) extended his earlier work on bounding the momentum transport in turbulent shear flows (Busse 1969*b*, referred to below as B69) to incorporate the continuity equation as an additional constraint. This improved bound still exceeded experimental measurements by an order of magnitude, but again, the optimizing flow field exhibited close similarities with the observed turbulent flow. Previous efforts to improve this correspondence have focused on maximizing a so-called 'efficiency function' (Ierley & Malkus 1988; Malkus & Smith 1989; Smith 1991) defined as the product of a drag coefficient and the ratio of the fluctuation and mean dissipation rate integrals. As in the case of maximum transport, the optimal solution is a discrete spectrum of streamwise vortices.

In this paper, we follow up an extension proposed by Busse (1978) and more recently rediscussed by Malkus & Smith (1989). Busse suggested that his earlier bound (B70) on the momentum transport for the turbulent shear flow might be tightened by adding a further constraint based on the poloidal power balance to the variational problem, which we formulate here in §2.1 using Lagrange multiplier methods. The chosen additional constraint is of particular interest because it forces the optimizing solution to become three-dimensional, even though experimental observations seem to confirm the salient features of Busse's two-dimensional multi-α solution (e.g. Townsend 1956). With this in mind, we construct trial functions in §2.2 for the variational problem with planforms which modulate Busse's streamwise rolls so that the motion exhibits weak three-dimensionality. An important ingredient is the presence of weak spanwise rolls, necessary to complete a potent triad interaction with the modulated streamwise rolls. By restricting our attention only to these solutions, we derive and solve a specialized form of the Euler–Lagrange equations in which the presence of higher-harmonic planforms is suppressed. These equations then define a simplified but nevertheless more tightly constrained optimization problem since the solution form is explicitly imposed. As a result, the calculated optimal momentum transport for this simpler problem provides a lower bound on the true maximal momentum transport attained over all allowable three-dimensional velocity fields. An upper bound on this true extremal value is provided by Busse's two-dimensional multi-α solution since this is generally believed to be the true optimizing solution to the original (B70) less-constrained variational problem where the poloidal power condition is not imposed. Hence we may 'sandwich' the true extremum of our problem, with the thickness of this 'sandwich' giving some idea of how effective the additional new constraint is in reducing the desired extremum. Indeed, since the 'sandwich' turns out to be so 'thin', it looks likely that our solution of the restricted problem also gives the lowest-order approximation to the more general problem. This is presumably because the true solution *is* a weakly three-dimensional solution with the excited higher harmonics having negligible effect.

In §3, we briefly summarize Busse's (B70) multi-α solution, linking it to our preferred Lagrange multiplier formulation. In §4, we incorporate the poloidal power constraint and develop an asymptotic analysis of our weakly three-dimensional solution as a small perturbation of Busse's along lines explained above. Interestingly, three-dimensionality is restricted to the outermost boundary layer and main stream. Order of magnitude estimates are then discussed which appear sufficient to assess the effect of adding the poloidal power constraint to the problem of maximizing the rate of dissipation in turbulent Couette flow. A few concluding remarks are added in §5.

2. Formulation

2.1. *The variational problem*

As in B70, we consider a homogeneous incompressible fluid of kinematic viscosity ν between two parallel, infinitely extended, rigid plates at $z = \pm\frac{1}{2}d$, which are sliding across each other with relative velocity V_0 in the constant direction $\widehat{\boldsymbol{x}}$. Using the plate separation distance d and the viscous diffusion timescale d^2/ν to non-dimensionalize the system leads to the Navier–Stokes equation

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V}\cdot\nabla\boldsymbol{V} + \nabla p = \nabla^2\boldsymbol{V} \tag{2.1}$$

with boundary condition

$$\boldsymbol{V} = \mp\tfrac{1}{2}Re\,\widehat{\boldsymbol{x}} \qquad \text{at } z = \pm\tfrac{1}{2},$$

where $Re := V_0 d/\nu$ is the Reynolds number. We make the standard assumption that the averages of the velocity components and their products over planes of constant z (denoted by an overbar) exist and are independent of time for a statistically steady turbulent flow. The total velocity $\boldsymbol{V}$ can then be separated into mean and fluctuating components as follows:

$$\boldsymbol{V}(\boldsymbol{x},t) \equiv \boldsymbol{U}(z) + \widehat{\boldsymbol{v}}(\boldsymbol{x},t)$$

with $\boldsymbol{U}(z) := \overline{\boldsymbol{V}(\boldsymbol{x},t)}$. Averaging the momentum equation (2.1) and integrating once then gives

$$\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}z} + Re\,\widehat{\boldsymbol{x}} + \langle\widehat{\boldsymbol{u}}\widehat{w}\rangle = \overline{\widehat{\boldsymbol{u}}\widehat{w}}, \tag{2.2}$$

where $\widehat{w}$ is the z-component of $\widehat{\boldsymbol{v}}$, $\widehat{\boldsymbol{u}}$ is the component parallel to the plates, and the angular brackets indicate the average over the entire fluid layer. It may be used to express the total power balance $\langle \boldsymbol{V}\cdot(2.1)\rangle$ in the form

$$\langle|\nabla\times\widehat{\boldsymbol{v}}|^2\rangle + \left\langle\left|\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}z}\right|^2\right\rangle = -Re\,\widehat{\boldsymbol{x}}\cdot\left.\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}z}\right|_{z=1/2} = Re^2 + Re\langle\widehat{u}\widehat{w}\rangle \qquad (\widehat{u} := \widehat{\boldsymbol{x}}\cdot\widehat{\boldsymbol{u}}),$$

where Re^2 is the dissipation corresponding to the laminar Couette solution. This implies that the quantity $Re\langle\widehat{u}\widehat{w}\rangle$ represents the advective part of the momentum transport between the plates. Further application of (2.2) to the total power balance equation gives

$$\langle|\nabla\times\widehat{\boldsymbol{v}}|^2\rangle + \langle|\overline{\widehat{\boldsymbol{u}}\widehat{w}} - \langle\widehat{\boldsymbol{u}}\widehat{w}\rangle|^2\rangle - Re\langle\widehat{u}\widehat{w}\rangle = 0$$

(B70 equation (2)).

At this point rather than maximizing $\mu := \langle uw\rangle$ at fixed Re, Busse chose the equivalent but more accessible problem of minimizing Re for fixed μ over the class of vector fields $\boldsymbol{v}$ which satisfy the boundary condition $\boldsymbol{v} = \mathbf{0}$ at $z = \pm\frac{1}{2}$, the continuity equation $\nabla \cdot \boldsymbol{v} = 0$ and the total power balance equation. The last constraint (B70 equation (2)) can then be used to efficiently reformulate the problem to one of minimizing the homogeneous functional

$$\mathscr{B}(\boldsymbol{v}, \mu) := \frac{\langle |\nabla \times \boldsymbol{v}|^2 \rangle}{\langle uw \rangle} + \mu \frac{\langle |\overline{\boldsymbol{u}w} - \langle \boldsymbol{u}w \rangle|^2 \rangle}{\langle uw \rangle^2}$$

(B70 equation (6)) over all solenoidal fields $\boldsymbol{v}$ which vanish on the plates.

In this paper, we consider Busse's problem of minimizing the Reynolds number Re at fixed momentum transport subject to an additional new constraint derived from the poloidal power balance. Introducing the toroidal–poloidal decomposition of the fluctuating field

$$\widehat{\boldsymbol{v}} := \widehat{\boldsymbol{v}}_t + \widehat{\boldsymbol{v}}_p := \nabla \times \psi \widehat{\boldsymbol{z}} + \nabla \times (\nabla \times v \widehat{\boldsymbol{z}}) \qquad (\langle \psi \rangle = \langle v \rangle = 0)$$

with $\psi = v = v_z = 0$ at $z = \pm\frac{1}{2}$, the poloidal and toroidal power balances are then just $\langle \widehat{\boldsymbol{v}}_p \cdot (2.1) \rangle$

$$\mathscr{D}_p = -\left\langle \sigma_p \frac{\mathrm{d}U}{\mathrm{d}z} \right\rangle + \Pi_p + \Pi_t$$

and $\langle \widehat{\boldsymbol{v}}_t \cdot (2.1) \rangle$

$$\mathscr{D}_t = -\left\langle \sigma_t \frac{\mathrm{d}U}{\mathrm{d}z} \right\rangle - \Pi_p - \Pi_t$$

(Busse 1978, Section VII.B), where we have assumed that $\boldsymbol{U} = U(z)\widehat{\boldsymbol{x}}$ and made the following definitions:

$$\left.\begin{aligned}
\sigma_p &:= u_p w = -v_{xz} \nabla_H^2 v, \\
\sigma_t &:= u_t w = -\psi_y \nabla_H^2 v, \\
\mathscr{D}_p &:= \langle |\nabla \widehat{\boldsymbol{v}}_p|^2 \rangle = \langle |\nabla_H \nabla^2 v|^2 \rangle, \\
\mathscr{D}_t &:= \langle |\nabla \widehat{\boldsymbol{v}}_t|^2 \rangle = \langle |\nabla_H \psi_z|^2 + |\nabla_H^2 \psi|^2 \rangle, \\
\Pi_p &:= \langle \widehat{\boldsymbol{v}}_t \cdot (\widehat{\boldsymbol{v}}_p \cdot \nabla) \widehat{\boldsymbol{v}}_p \rangle = \langle \nabla_H^2 v \{ v_{yzz} \psi_x - v_{xzz} \psi_y \} \rangle, \\
\Pi_t &:= \langle \widehat{\boldsymbol{v}}_t \cdot (\widehat{\boldsymbol{v}}_t \cdot \nabla) \widehat{\boldsymbol{v}}_p \rangle = \langle 2 v_z \{ \psi_{xy}^2 - \psi_{xx} \psi_{yy} \} \rangle
\end{aligned}\right\} \tag{2.3}$$

with $\nabla_H := \widehat{\boldsymbol{x}}\partial_x + \widehat{\boldsymbol{y}}\partial_y$. These used in conjunction with the mean equation (2.2) then lead to the normal total power relation

$$\mu Re = \mathscr{D}_p + \mathscr{D}_t + \langle |\overline{\sigma} - \langle \sigma \rangle|^2 \rangle \qquad (\mu \equiv \langle \sigma \rangle), \tag{2.4}$$

where $\sigma := \sigma_p + \sigma_t$, together with the new poloidal power relation

$$Re \langle \sigma_p \rangle + \Pi_p + \Pi_t = \mathscr{D}_p + \langle (\overline{\sigma}_p - \langle \sigma_p \rangle)(\overline{\sigma} - \langle \sigma \rangle) \rangle. \tag{2.5}$$

With this additional constraint, the minimization of Re at fixed μ cannot now be recast in favour of a homogeneous functional and we are forced to construct the full

Lagrangian

$$\begin{aligned}\mathscr{L} := \mu^{-1}\ \{\ &\mathscr{D}_p + \mathscr{D}_t + \langle|\overline{\sigma} - \langle\sigma\rangle|^2\rangle \\ &+ \gamma[\mathscr{D}_p + \langle(\overline{\sigma}_p - \langle\sigma_p\rangle)(\overline{\sigma} - \langle\sigma\rangle)\rangle - Re\langle\sigma_p\rangle - \Pi_p - \Pi_t] \\ &+ \lambda[\mu - \langle\sigma\rangle]\},\end{aligned} \tag{2.6}$$

where γ and λ are Lagrange multipliers.

2.2. *The trial functions*

The poloidal power constraint (2.4) has two major effects on the subsequent optimal problem to be solved. Firstly, the constraint cannot be met by two-dimensional flows independent of the streamwise coordinate x, because then all the terms in (2.5) vanish except for the positive term $\mathscr{D}_p$. This means that Busse's multi-α (streamwise rolls) solution (B70) fails to satisfy the constraint in a fundamental way and the optimal solution is forced to be three-dimensional. Secondly, the cubically nonlinear term $\Pi_p + \Pi_t$ introduces a new quadratic nonlinearity into the Euler–Lagrange problem, potentially changing its whole character. Evidence that it, in fact, does not is provided by experimental observations which appear to confirm the salient features of Busse's streamwise rolls. As a result, we search for a slightly perturbed three-dimensional version of Busse's multi-α solution to our optimal problem. The simplest such is based upon a symmetric triad in wavenumber space as follows:

$$\left.\begin{aligned} v &= \mu^{1/2}\sum_{n=1}^{N}\left\{\frac{1}{k_n^2 + l_n^2}\phi_n(x,y)w_n(z) + \frac{1}{4l_n^2}\Phi_n(x)W_n(z)\right\}, \\ \psi_y &= \mu^{1/2}\sum_{n=1}^{N}\phi_n(x,y)\theta_n(z), \end{aligned}\right\} \tag{2.7}$$

where

$$\phi_n(x,y) := \cos(k_n y + l_n x) + \cos(k_n y - l_n x) \equiv 2\cos(k_n y)\cos(l_n x)$$

defines a rectangular planform of large aspect ratio and

$$\Phi_n(x) := \sqrt{2}\cos(2l_n x) \qquad (\, l_n \ll k_n\,)$$

identifies the planform for the new spanwise rolls; they have been normalized so that $\langle\phi_n^2\rangle = \langle\Phi_n^2\rangle = 1$. For this ansatz, the poloidal component of the stress vanishes, as does one of the nonlinear terms. Consequently, the poloidal power relation (2.5) simplifies to just

$$\Pi_t = \mathscr{D}_p \qquad (\overline{\sigma}_p = \Pi_p = 0), \tag{2.8}$$

and the Lagrangian becomes

$$\mathscr{L} = \lambda + \mu\left\langle\left(1 - \sum_{n=1}^{N}\sigma_n\right)^2\right\rangle + \sum_{n=1}^{N}\left[(1+\gamma)\left(\mathscr{D}_w^{(n)} + \mathscr{D}_W^{(n)}\right) + \mathscr{D}_\theta^{(n)} - \lambda\langle\sigma_n\rangle - \gamma\mu^{1/2}\Pi_n\right], \tag{2.9}$$

where

$$\left.\begin{aligned}
&\bar{\sigma}=\bar{\sigma}_t=\mu\sum_{n=1}^{N}\sigma_n, && \sigma_n := w_n\theta_n,\\
&\Pi_t=\mu^{3/2}\sum_{n=1}^{N}\Pi_n, && \Pi_n := -\frac{1}{\sqrt{2}}\left\langle\frac{\mathrm{d}W_n}{\mathrm{d}z}\theta_n^2\right\rangle,\\
&\mathscr{D}_p=\mu\sum_{n=1}^{N}\{\mathscr{D}_w^{(n)}+\mathscr{D}_W^{(n)}\}, && \mathscr{D}_t=\mu\sum_{n=1}^{N}\mathscr{D}_\theta^{(n)},\\
&\mathscr{D}_w^{(n)} := (k_n^2+l_n^2)\langle|w_n|^2\rangle+2\left\langle\left|\frac{\mathrm{d}w_n}{\mathrm{d}z}\right|^2\right\rangle+\frac{1}{k_n^2+l_n^2}\left\langle\left|\frac{\mathrm{d}^2w_n}{\mathrm{d}z^2}\right|^2\right\rangle,\\
&\mathscr{D}_W^{(n)} := 4l_n^2\langle|W_n|^2\rangle+2\left\langle\left|\frac{\mathrm{d}W_n}{\mathrm{d}z}\right|^2\right\rangle+\frac{1}{4l_n^2}\left\langle\left|\frac{\mathrm{d}^2W_n}{\mathrm{d}z^2}\right|^2\right\rangle,\\
&\mathscr{D}_\theta^{(n)} := \frac{k_n^2+l_n^2}{k_n^2}\left\{(k_n^2+l_n^2)\langle|\theta_n|^2\rangle+\left\langle\left|\frac{\mathrm{d}\theta_n}{\mathrm{d}z}\right|^2\right\rangle\right\}.
\end{aligned}\right\}\tag{2.10}$$

The corresponding Euler–Lagrange equations are

$$\delta w_n:\ \frac{(1+\gamma)}{k_n^2+l_n^2}\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}-(k_n^2+l_n^2)\right]^2 w_n-\left\{\tfrac{1}{2}\lambda+\mu\left(1-\sum_{n=1}^{N}\sigma_n\right)\right\}\theta_n=0,\tag{2.11}$$

$$\delta\theta_n:\ -\frac{k_n^2+l_n^2}{k_n^2}\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}-(k_n^2+l_n^2)\right]\theta_n-\left\{\tfrac{1}{2}\lambda+\mu\left(1-\sum_{n=1}^{N}\sigma_n\right)\right\}w_n=-\frac{\gamma\mu^{1/2}}{\sqrt{2}}\frac{\mathrm{d}W_n}{\mathrm{d}z}\theta_n,\tag{2.12}$$

$$\delta W_n:\ \frac{(1+\gamma)}{4l_n^2}\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}-4l_n^2\right]^2 W_n=\frac{\gamma\mu^{1/2}}{\sqrt{2}}\theta_n\frac{\mathrm{d}\theta_n}{\mathrm{d}z}\tag{2.13}$$

to be solved subject to the boundary conditions

$$w_n=\frac{\mathrm{d}w_n}{\mathrm{d}z}=W_n=\frac{\mathrm{d}W_n}{\mathrm{d}z}=\theta_n=0\qquad\text{on } z=\pm\tfrac{1}{2}.\tag{2.14}$$

Minimization with respect to variations of k_n yields

$$\begin{aligned}\delta k_n:\quad &(1+\gamma)\left\{(k_n^2+l_n^2)\langle|w_n|^2\rangle-\frac{1}{k_n^2+l_n^2}\left\langle\left|\frac{\mathrm{d}^2w_n}{\mathrm{d}z^2}\right|^2\right\rangle\right\}\\ &+\frac{k_n^2+l_n^2}{k_n^2}\left\{\frac{k_n^4-l_n^4}{k_n^2}\langle|\theta_n|^2\rangle-\frac{l_n^2}{k_n^2}\left\langle\left|\frac{\mathrm{d}\theta_n}{\mathrm{d}z}\right|^2\right\rangle\right\}=0,\end{aligned}\tag{2.15}$$

while in the case of l_n there are two possibilities: either $l_n=0$ with $l_n^{-2}W_n=0$, or

$l_n \neq 0$ with

$$\delta l_n: \quad (1+\gamma)\left\{ (k_n^2+l_n^2)\langle |w_n|^2\rangle - \frac{1}{k_n^2+l_n^2}\left\langle \left|\frac{\mathrm{d}^2 w_n}{\mathrm{d}z^2}\right|^2 \right\rangle \right.$$
$$\left. + \frac{k_n^2+l_n^2}{l_n^2}\left[4l_n^2\langle |W_n|^2\rangle - \frac{1}{4l_n^2}\left\langle \left|\frac{\mathrm{d}^2 W_n}{\mathrm{d}z^2}\right|^2 \right\rangle\right]\right\}$$
$$+ \frac{k_n^2+l_n^2}{k_n^2}\left[2(k_n^2+l_n^2)\langle |\theta_n|^2\rangle + \left\langle \left|\frac{\mathrm{d}\theta_n}{\mathrm{d}z}\right|^2 \right\rangle\right] = 0. \tag{2.16}$$

Subtracting (2.16) from (2.15) gives

$$(1+\gamma)\left\{\frac{1}{4l_n^2}\left\langle \left|\frac{\mathrm{d}^2 W_n}{\mathrm{d}z^2}\right|^2 \right\rangle - 4l_n^2\langle |W_n|^2\rangle\right\} = \frac{l_n^2(k_n^2+l_n^2)}{k_n^4}\left\{(k_n^2+l_n^2)\langle |\theta_n|^2\rangle + \left\langle \left|\frac{\mathrm{d}\theta_n}{\mathrm{d}z}\right|^2 \right\rangle\right\}. \tag{2.17}$$

We now look for a solution with (i) $\gamma \ll 1$ and (ii) $l_n^2 \ll k_n^2$ so that Busse's solution is recovered at leading order.

3. Busse's multi-α solution

When only the total power integral (2.4) is imposed and the poloidal power integral (2.5) is ignored, we set

$$\gamma = 0$$

in (2.6) and recover Busse's (B70) problem but formulated with the additional Lagrange multiplier λ. The link between the formulations is forged from the power integrals of the Euler–Lagrange equations, namely the vanishing of the functional derivatives $\partial\mathscr{L}/\partial v$ and $\partial\mathscr{L}/\partial\psi$. They lead to the result

$$\tfrac{1}{2}\mu\left(\left\langle v\frac{\partial\mathscr{L}}{\partial v}\right\rangle + \left\langle \psi\frac{\partial\mathscr{L}}{\partial\psi}\right\rangle\right) = -\mu\lambda + \mathscr{D}_p + \mathscr{D}_t + 2\langle|\overline{\sigma} - \langle\sigma\rangle|^2\rangle = 0,$$

which with (2.4) yields the explicit formula

$$\mu(\lambda - Re) = \langle|\overline{\sigma} - \langle\sigma\rangle|^2\rangle \tag{3.1}$$

for λ.

The B70 optimizing solution is of the form (2.7) but two-dimensional with

$$l_n = 0 \qquad \text{and} \qquad l_n^{-2}W_n = 0$$

for all n. It consists of N boundary layers (identified by each of the N harmonics) in which the appropriate stretched length is

$$\xi_n := (k_{n+1}^2 k_n)^{1/3}(\tfrac{1}{2} \mp z);$$

here the dummy wavenumber $k_{N+1} := \mu^{1/2}$ is introduced to simplify the notation. The nth harmonic has a triple-deck structure. Its inner boundary layer deck, which defines the nth layer $(\xi_n = O(1))$, has solution

$$\widehat{w}_n(z) = \delta_n^{1/2} \begin{cases} \widehat{\Omega}(\xi_n), & n = 1, \ldots, N-1 \\ \Omega(\xi_N), & n = N, \end{cases}$$

$$\widehat{\theta}_n(z) = \delta_n^{-1/2} \begin{cases} \widehat{\Theta}(\xi_n), & n = 1, \ldots, N-1 \\ \Theta(\xi_N), & n = N, \end{cases}$$

in which

$$\delta_n = \left(k_n/k_{n+1}\right)^{2/3}$$

is the small aspect ratio of vertical-z to horizontal-x length scales. The solution in the intermediate deck, where the aspect ratio is $O(1)$, makes no significant contribution to the optimization problem and so is not considered. The outer deck of the first harmonic ($n = 1$) coincides with the main stream $\left(\frac{1}{2} - |z| = O(1)\right.$ referred to as the *zero*th layer$\left.\right)$, while those for the higher harmonics ($2 \leqslant n \leqslant N$) lie in the $(n-1)$th layer and have large aspect ratio $\delta_{n-1}^{-1/2}$. The outer-deck solution for the $(n+1)$th harmonic in the nth layer is

$$\widetilde{w}_{n+1}^2 = \widetilde{\theta}_{n+1}^2 = \begin{cases} 1, & n = 0 \\ 1 - \widehat{\Omega}(\xi_n)\widehat{\Theta}(\xi_n), & n = 1, \ldots, N-1. \end{cases}$$

The intriguing feature of Busse's multi-α solution (in our notation multi-k!) is the way in which the triple deck of the nth harmonic interweaves with its neighbouring $(n+1)$th harmonic in order that the momentum transport remains close to its mean value in all layers except the wall boundary layer ($n = N$). With the definition

$$G(\xi_n) := k_{n+1}^{-2}\, \mu(1 - \widetilde{w}_{n+1}\widetilde{\theta}_{n+1} - \widehat{w}_n\widehat{\theta}_n), \qquad n = 1, \ldots, N-1,$$

where $G(\xi_n)$ is an $O(1)$ function whose properties are defined in the Appendix, the value of the mean velocity gradient $\mathrm{d}U/\mathrm{d}z$ in the nth layer is

$$\left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)_n = \begin{cases} -Re + \frac{1}{2}\lambda - k_1^2, & n = 0 \\ -k_{n+1}^2 G(\xi_n), & n = 1, \ldots, N-1 \\ -\mu\left(1 - \Omega(\xi_N)\Theta(\xi_N)\right), & n = N. \end{cases}$$

The corresponding jump in mean velocity from the mid-plane ($z = 0$) to the upper boundary $\left(z = \frac{1}{2}\right)$ across each layer is

$$\Delta U_n := \begin{cases} \displaystyle\int_0^{1/2} \left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)_0 \mathrm{d}z, & n = 0 \\ \displaystyle(k_{n+1}^2 k_n)^{-1/3} \int_0^{\infty} \left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)_n \mathrm{d}\xi_n, & n = 1, \ldots, N. \end{cases}$$

From the solution of the Euler–Lagrange equations, Busse (B70) determined an expression for Re, which he minimized with respect to each of the k_n. In our formulation an equivalent procedure is the use of (2.15). In terms of

$$k_n^2 := \mu^{q_n} b_n^2, \qquad q_n := \frac{2 - 4^{-n+1}}{2 - 4^{-N}}, \tag{3.2}$$

either method leads to Busse's formula

$$\left(\frac{b_2^4}{b_1}\right)^{1/3} = \frac{b_1^2}{\beta},$$
$$\left(\frac{b_{n+1}^4}{b_n}\right)^{1/3} = 4\left(\frac{b_n^4}{b_{n-1}}\right)^{1/3}\begin{cases} 1, & n = 2,\dots,N-1 \\ (\sigma/\beta)^{-1}, & n = N, \end{cases}$$
$$b_{N+1} = 1$$

(B70 above equation (29)), where

$$\sigma := \int_0^\infty \Omega''^2 \mathrm{d}\xi_N \equiv \int_0^\infty \Theta'^2 \mathrm{d}\xi_N \equiv \tfrac{1}{5}\int_0^\infty (1-\Omega\Theta)\,\mathrm{d}\xi_N \equiv \tfrac{1}{4}\int_0^\infty (1-\Omega\Theta)^2 \mathrm{d}\xi_N \approx 0.337$$

(B69 equation (3.27)),

$$\beta := \int_0^\infty \widehat{\Omega}''^2 \mathrm{d}\xi_n \equiv \int_0^\infty \widehat{\Theta}'^2 \mathrm{d}\xi_n \equiv \tfrac{1}{3}\int_0^\infty G\,\mathrm{d}\xi_n \equiv \tfrac{1}{2}\int_0^\infty (1-\widehat{\Omega}\widehat{\Theta})\,\mathrm{d}\xi_n \approx 0.624$$

(B69 equation (A7) and below; the final identity was quoted by Busse (1978) between his equations (4.17) and (4.18) and we give its derivation in the Appendix via the key equation (A 4)). We rewrite the solution of the nonlinear recurrence relations (B70 equation (29)) in favour of the parameter

$$\Upsilon_N := 4^{-2N-8/3}\beta^{-2}(\sigma/\beta)^{3/2}, \tag{3.3}$$

which gives

$$k_n = \beta 4^{n+1/3}(\mu\Upsilon_N)^{\frac{1}{2}q_n}\begin{cases} 1, & n = 1,\dots,N \\ (\sigma/\beta)^{-3/4}, & n = N+1 \end{cases} \quad (\text{i.e.} \quad k_{N+1} = \mu^{1/2}). \tag{3.4}$$

From them we may express our aspect ratios in the form

$$\delta_n = 4^{-2/3}(\mu\Upsilon_N)^{-2p_n}\begin{cases} 1, & n = 1,\dots,N-1 \\ (\sigma/\beta)^{1/2}, & n = N, \end{cases} \tag{3.5}$$

where

$$p_n := \tfrac{1}{6}(q_{n+1} - q_n) \equiv \frac{4^{-n}}{2(2-4^{-N})}. \tag{3.6}$$

As $\delta_1 \ll \delta_2 \ll \dots \ll \delta_N$, the scale separations necessary for the validity of the asymptotics only occur when

$$\delta_N = \left(k_N^2/\mu\right)^{1/3} = 4^{-2/3}(\sigma/\beta)^{1/2}(\mu\Upsilon_N)^{-4^{-N}/(2-4^{-N})} \tag{3.7}$$

is small.

It should be emphasized that, whereas the contributions to the poloidal and toroidal dissipations $\mathscr{D}_p$ and $\mathscr{D}_t$ are comparable in each layer, it is only in the Nth layer adjacent to the wall that there is a significant contribution to the mean flow dissipation $\langle|\mathrm{d}U/\mathrm{d}z|^2\rangle$. So to leading order, we have

$$\left\langle|\overline{\sigma} - \langle\sigma\rangle|^2\right\rangle = 2\delta_N^{-1/2}\mu^{3/2}\int_0^\infty (1-\Omega\Theta)^2 \mathrm{d}\xi_N. \tag{3.8}$$

It is the same order of magnitude as μRe and so both terms must be retained in the balance (3.1) for λ. In this way equations (3.1), (3.8), (2.11) and (2.12) reproduce B70 equations (21), (22) and (23), for which the solution is

$$\langle|\bar{\sigma}-\langle\sigma\rangle|^2\rangle = 8\sigma\delta_N^{-1/2}\mu^{3/2}, \qquad \mathscr{D}_t = \mathscr{D}_p = 4(1-4^{-N})\sigma\delta_N^{-1/2}\mu^{3/2}. \tag{3.9}$$

Together with (2.4) these yield the result

$$Re = 8(2-4^{-N})\sigma\delta_N^{-1/2}\mu^{1/2} \tag{3.10}$$

(B70 equation (30)). The jumps in mean velocity from the mid-plane to the upper boundary are

$$\Delta U_n = -\sigma\delta_N^{-1/2}\mu^{1/2}\begin{cases} 2, & n=0 \\ 3.4^{n-N}, & n=1,\ldots,N-1 \\ 5, & n=N \end{cases} \tag{3.11}$$

consistent with the requirement that

$$Re = -2\sum_{n=0}^{N}\Delta U_n.$$

In the limit $Re \uparrow \infty$ the best bounds are achieved when N is large:

$$N \gg 1,$$

though, as we explain below, some care must be taken with the double limit in which $N \uparrow \infty$ as well. For large N, the results (3.9) take on the simple form

$$\tfrac{1}{2}\langle|\sigma-\langle\sigma\rangle|^2\rangle = \mathscr{D}_p = \mathscr{D}_t = \tfrac{1}{4}\mu Re, \tag{3.12}$$

while the mean flow velocity jumps across each layer are particularly illuminating:

$$-2\Delta U_0 = \tfrac{1}{4}Re, \qquad -2\sum_{n=1}^{N-1}\Delta U_n = \tfrac{1}{8}Re, \qquad -2\Delta U_N = \tfrac{5}{8}Re. \tag{3.13}$$

In other words, as Busse (B70) explained, a mean shear $U = -\frac{1}{4}Re\, z$ is maintained in the main stream ($n=0$). The main adjustment of U to the upper wall value $-\frac{1}{2}Re$ takes place in the wall layer ($n=N$) so that only a relatively small adjustment occurs in the outer layers ($1 \leqslant n \leqslant N-1$). Indeed the contributions drop off rapidly from one layer to the next by the factor a quarter with decreasing n.

Busse (B70) also poses the question "What is the optimum choice of N when $\mu \uparrow \infty$?" Minimizing Re as a function of N (regarded as a continuous variable) leads to that value of $N = N_o = \frac{1}{2}(\ln\mu/\ln 4) + O(1)$, which solves

$$\left(\mu \Upsilon_N\right)_{\{N=N_o\}} = 1 \tag{3.14}$$

(B70 equation (33)). Our representation (3.4) for k_n in terms of $\mu\Upsilon_N$ and the resulting expression (3.5), which determines the aspect ratio δ_n, highlight the fact that, in the limit (3.14), the necessary scale separation for the validity of the asymptotics is not strictly met. According to (3.5) the wavenumber of each successive boundary layer only increases by a factor 4:

$$k_{n+1}/k_n = 4, \qquad \delta_n = 4^{-2/3} \qquad n = 1,\ldots,N_o-1$$

(B70 equation (34)), while

$$\delta_{No} := \left(\delta_N\right)_{\{N=N_o\}} = 4^{-2/3}(\sigma/\beta)^{1/2} \approx 0.292. \tag{3.15}$$

Consequently, (3.10) yields

$$Re = Re_o \equiv 4^{7/3}(\sigma^3\beta)^{1/4}\mu^{1/2} \tag{3.16}$$

(B70 equation (31)). The result is very suggestive because of the simple power laws implied. Of course, in any physically realizable situation μ though large is necessarily finite and the corresponding optimizing N is then only moderately large.

Since δ_{No} is $O(1)$ and not small, the asymptotic approximations are not justified as N increases up to N_o. We have, therefore, been careful to highlight the properties (3.12) and (3.13) of the dissipation rates and mean flow U respectively, which hold for large N in the valid limit

$$\delta_N \ll 1 \qquad (\mu\Upsilon_N \gg 1, \;\; N \gg 1). \tag{3.17}$$

This is probably why the linear mean flow profile in the main stream predicted by (3.13) agrees so well with the experimental results (B70). The attractive feature of the bounding method is that we are not forced to take the value $N = N_o$; instead, we are free to fix N and ask about the nature of the results as $Re \uparrow \infty$ consistent with the inequality (3.17). Within the framework of that class of trial functions, the asymptotically correct large-N solution is

$$Re = \left(\delta_{N_o}/\delta_N\right)^{1/2} Re_o, \tag{3.18}$$

where

$$\delta_{N_o}/\delta_N = 4^{(N_o-N)4^{-N}} \;(\gg 1),$$

valid when

$$1 \ll N \ll \ln N_o = O\big(\ln(\ln\mu)\big).$$

These limits on N are extremely restricting and we may only speculate on how large we can push N to obtain physically reasonable results. Indeed, the boundary layer aspect ratios $\delta_n = 4^{-2/3}$ ($n = 1, \ldots, N_o - 1$) appropriate to the case $N = N_o$ are conceivably small enough. A plausible scenario is that (3.18) gives the correct bound with Re proportional to $\mu^{1/2}$ (via Re_o), but that the coefficient $\left(\delta_{N_o}/\delta_N\right)^{1/2}$ varies slowly with μ possibly with a natural functional dependence on $\ln\mu$. This would be compatible with experimental results over a finite range, which revealed a slope 2 for plots of $\log\mu$ versus $\log Re$.

Of course, the subtleties of the limiting procedures continue to plague the three-dimensional solution presented in the following section. Furthermore, we again only consider a limited set of trial functions but the new issues raised are of a different type, the point being to show that the addition of the poloidal power constraint does not alter the bound presented here at leading order.

4. Poloidal power constraint

When we take account of the poloidal power integral (2.5), we may derive an explicit formula for the Lagrange multiplier γ as in (3.1) by forming the difference (rather than the sum)

$$\tfrac{1}{2}\mu\left(\left\langle v\frac{\partial\mathscr{L}}{\partial v}\right\rangle - \left\langle \psi\frac{\partial\mathscr{L}}{\partial\psi}\right\rangle\right) = (\mathscr{D}_p - \mathscr{D}_t) + \gamma(\mathscr{D}_p + \tfrac{1}{2}\Pi_t) + \ldots = 0,$$

where the terms neglected "..." vanish with our ansatz for which $\bar{\sigma}_p = \Pi_p = 0$. Using (2.8), it gives

$$\mathscr{D}_t - \mathscr{D}_p = \tfrac{3}{2}\gamma\mathscr{D}_p + \dots . \tag{4.1}$$

Whereas in Busse's (B70) problem the toroidal and poloidal dissipations are equal, in our problem, γ provides a measure of the departure from that equality.

With $0 < \gamma \ll 1$, the method of solution proceeds through consideration of equations (2.8), (2.13) and (2.17). The Euler–Lagrange equation with respect to W_n, (2.13), can be viewed as an inner equation for $\widehat{W}_n$ forced by Busse's multi-α solution $\widehat{\theta}_n$ over the nth layer $(\frac{1}{2} \mp z) = O\big((k_{n+1}^2 k_n)^{-1/3}\big)$, together with an unforced outer equation for $\widetilde{W}_n$ when $(\frac{1}{2} \mp z) = O(l_n^{-1})$. Assuming that $\mathrm{d}\widehat{W}_n/\mathrm{d}z = O\big(\widehat{W}_n(k_{n+1}^2 k_n)^{1/3}\big)$ at the edge of the inner region, the velocity in the outer region can be estimated simply as $\widetilde{W}_n = O\big(\widehat{W}_n(k_{n+1}^2 k_n)^{1/3}/l_n\big)$, implying that

$$\frac{\langle l_n^2 \widetilde{W}_n^2\rangle}{\left\langle l_n^{-2}\left|\mathrm{d}^2\widehat{W}_n/\mathrm{d}z^2\right|^2\right\rangle} \sim O\left(\frac{l_n}{k_n}\,\delta_n\right) \ll 1.$$

Hence the dissipation associated with W_n is essentially contained within the inner region. Furthermore, it will transpire that $(k_{n+1}^2 k_n)^{-1/3} \ll k_n^{-1} \ll l_n^{-1} \ll (k_n^2 k_{n-1})^{-1/3}$ so that $\widetilde{W}_n$ does not interfere with the next boundary layer. As a result, the outer solution $\widetilde{W}_n$ need not be discussed further.

Equation (2.13) determines $\widehat{W}_n/l_n^2$ uniquely for each layer n modulo an overall amplitude dependent on γ. Independent of this, equation (2.17) implies an integral constraint on $\widehat{W}_n/l_n^2$ for each layer which, in general, will be inconsistent with the solutions determined by (2.13). As a result $l_n = 0$ is forced in all but one layer for which the value of γ may be chosen to allow consistency. Hence we have immediately that within our ansatz, streamwise variation will arise in only one triple deck. Guided by our earlier representation for $\widehat{w}_n(z)$ in its inner deck, we cast the new inner solution in the form

$$\widehat{W}_n(z) = (2\delta_n)^{1/2} A_n \widehat{W}(\xi_n), \tag{4.2}$$

where A_n is a constant that can be defined at our convenience. Our choice and appropriate scalings are as follows:

$$A_n \equiv \mu^{-t_n} a_n, \qquad \frac{l_n^2}{k_n^2} := \mu^{-t_n} c_n^2, \qquad \gamma := \mu^{-t_n}\frac{b_{n+1}a_n}{c_n^2}.$$

Here the power laws involving

$$t_n := p_n - p_N \equiv \frac{4^{-n} - 4^{-N}}{2(2-4^{-N})} \tag{4.3}$$

are significant in so much as they relate to the ratio

$$\left(\frac{\delta_n}{\delta_N}\right)^{1/2} = \left(\mu \Upsilon_N\right)^{-t_n}\begin{cases} (\sigma/\beta)^{-1/4}, & n = 1, \dots, N-1 \\ 1, & n = N. \end{cases}$$

Within the framework of the boundary layer approximations, two integrations of equation (2.13) yields

$$\widehat{W}'' = -\int_{\xi}^{\infty} \widehat{\Theta}^2 \mathrm{d}\xi. \tag{4.4}$$

The actual value of $\widehat{W}$ follows from two further integrations subject to

$$\widehat{W}(0) = \widehat{W}'(0) = 0.$$

All that is required from this system is the $O(1)$ quantity

$$\rho := \int_0^\infty \widehat{W}''^2 \mathrm{d}\xi \equiv -\int_0^\infty \widehat{W}'\widehat{\Theta}^2 \mathrm{d}\xi. \tag{4.5}$$

Note that these integrals are convergent, consistent with our earlier estimate that the dissipation in the outer region is negligible, because $\widehat{W}''$ like $\widehat{\Theta}$ is of order ξ^{-1} as $\xi \uparrow \infty$.

The amplitude A_n of the spanwise roll is fixed by the poloidal power constraint (2.8) which gives

$$\Pi_n = 2\rho\delta_n^{-1/2}A_n = \mu^{-3/2}\mathscr{D}_p, \tag{4.6}$$

where $\mathscr{D}_p$ is given correct to leading order by (3.9). In turn, its wavenumber $2l_n$ is fixed by the minimization condition (2.17) yielding

$$\frac{a_n}{c_n^2} = \left[\frac{3\beta}{\rho}\right]^{1/2} \begin{cases} 1, & n = 1, \dots, N-1 \\ (\sigma/\beta)^{1/2}, & n = N. \end{cases} \tag{4.7}$$

With b_n given at leading order by (3.2) and (3.4) we solve successively for a_n and c_n to obtain

$$A_n = 2(1-4^{-N})\frac{\sigma}{\rho}\left(\frac{\delta_n}{\delta_N}\right)^{1/2}, \tag{4.8}$$

$$\frac{l_n^2}{k_n^2} = 2(1-4^{-N})\left[\frac{\sigma^2}{3\beta\rho}\right]^{1/2}\left(\frac{\delta_n}{\delta_N}\right)^{1/2} \begin{cases} 1, & n = 1, \dots, N-1 \\ (\sigma/\beta)^{-1/2}, & n = N, \end{cases} \tag{4.9}$$

$$\gamma = 4^{n-N}\left[\frac{3\sigma^2}{\beta\rho}\right]^{1/2}\left(\frac{\delta_n}{\delta_N}\right)^{1/2} \begin{cases} 1, & n = 1, \dots, N-1 \\ (\sigma/\beta)^{-1/2}, & n = N. \end{cases} \tag{4.10}$$

From an asymptotic point of view, the solution with $n = N$ is unacceptable, because t_N is zero with the consequence that both γ and l_N/k_N are of order unity, as is clear from (4.10) and (4.9). This means, for example, that our results are not applicable to Howard's (1963) ($N = 1$) problem for which the only available choice is $n = 1$ also. Our results suggest that the introduction of the poloidal power integral constraint (2.5) leads to a significant $O(1)$ change in Howard's result, but the precise modifications to his results cannot be predicted because of the failure of our asymptotic approximations.

For fixed $N > 1$ and $n < N$, however, this large-μ solution (4.8)–(4.10) is consistent with our initial assumptions that (i) $\gamma \ll 1$ and (ii) $l_n^2 \ll k_n^2$, hence justifying the regular perturbative approach adopted. The dissipation $\mathscr{D}_W^{(n)}$ associated with the spanwise roll component is easily found by combining (2.8) with $\langle W_n(2.13)\rangle$ to be $O(\gamma)$ relative to the dissipation associated with the streamwise rolls:

$$(1+\gamma)\mathscr{D}_W^{(n)} = \tfrac{1}{2}\gamma\mu^{1/2}\Pi_n = \tfrac{1}{2}\gamma\mu^{-1}\mathscr{D}_p. \tag{4.11}$$

The corrections to w_n and θ_n are also $O(\gamma)$ leading to $O(\gamma)$ modifications in $\mathscr{D}_\theta^{(n)}$ and $\mathscr{D}_w^{(n)}$ from their B70 values consistent with (4.1). All this leads to the conclusion that the minimum Reynolds number Re_{ks} determined by our theory exceeds Busse's value

Re_b by an amount of $O(\gamma)$:

$$Re_{ks} = Re_b + O(\gamma). \tag{4.12}$$

In other words, at fixed Reynolds number, the reduction in the extremal dissipation rate is $O(\gamma)$ (this may be formalized at the considerable expense of solving equations (2.11) and (2.12) for the $O(\gamma)$ corrections to w_n and θ_n). Since γ is minimized (in order of magnitude) when $n = 1$, the minimum penalty for satisfying the new poloidal power constraint is exacted by placing the three-dimensionality in the uppermost layer. Here, evidently, the disruption to Busse's two-dimensional multi-α solution is felt least.

Though there is no guarantee that the perturbative trial solution discussed above will supply the true *global* minimum value of *Re* for our variational problem, it seems likely, however, that it does. That happens provided that the excitation of higher planform harmonics can be neglected in the lowest-order approximations to the complete Euler–Lagrange equations $\partial\mathscr{L}/\partial\psi = 0$ and $\partial\mathscr{L}/\partial v = 0$, which certainly appears to be the case. For our purposes, that tighter result is unnecessary; all that we need is the property that our solution necessarily supplies an upper bound on this minimum Reynolds number. A lower bound is provided by Busse's (B70) result assuming that his multi-α solution really is, as conjectured, the optimizing solution for his less constrained problem. As a result, we can sandwich the true extremum of the dissipation rate for our problem between two estimates which agree to leading order in the Reynolds number. It is then clear that the imposition of the poloidal power constraint leaves Busse's upper bound for the momentum transport in turbulent Couette flow essentially unchanged.

When like B70 we take the optimum $N = N_o$, our results (4.9) and (4.10) become

$$\frac{l_n^2}{k_n^2} = 2\left[\frac{\sigma^3}{9\beta\rho^2}\right]^{1/4} \begin{cases} 1, & n = 1,\ldots,N_o - 1 \\ (\sigma/\beta)^{-1/4}, & n = N_o, \end{cases} \tag{4.13}$$

$$\gamma = 4^{n+4/3}\mu^{-1/2}\left[\frac{3\beta^3}{\rho}\right]^{1/2} \begin{cases} 1, & n = 1,\ldots,N_o - 1 \\ (\sigma/\beta)^{-1/4}, & n = N_o \end{cases} \tag{4.14}$$

as $\mu \uparrow \infty$. Reassuringly, the optimal value (4.14) of γ is small. On the other hand, the ratio (4.13) of the squared wavenumbers is of order unity and not small as required. Of course, this result is to be expected in view of the failure of the Busse solution to achieve proper scale separation in this limit. The actual numerical value of the ratio (4.13) depends on ρ, which we may estimate to be roughly β^2; its exact value, which we do not need, may be calculated from the formulae in B69, Appendix A.

5. Discussion

In this paper, through the restriction of our trial functions to simple planforms, we have established that the inclusion of the poloidal power balance as a new constraint on Busse's original variational problem (B70) has no effect to leading order on the optimal dissipation rate. Busse's optimizing solution merely adapts in the least disruptive way to the new constraint by adjusting its structure solely in the outermost boundary layer. The point is that the necessary size of the vital cubic interaction Π_t in (2.5) and consequently the required amplitude of the spanwise roll component in each boundary layer is fixed by Busse's solution. According to (4.11), the dissipation associated with the spanwise roll is then smallest if that roll is located in the outermost boundary layer. Encouragingly, the emergence of spanwise roll structure in the outer regions of the boundary layers is seen in experiments. Kim, Kline & Reynolds (1971), for example, discuss the appearance of spanwise vortices at the edge of the turbulent boundary layer over a flat plate.

More generally, the ease with which Busse's multi-α solution adjusts to the poloidal power constraint with minimal cost to the extremum does not auger well for the prospects of other as-yet-untried 'global' constraints. It may well be that the only way forward to improve Busse's result is to impose pointwise constraints across the channel width, forcing adjustment throughout the whole boundary layer structure. However, whether such constraints are available and any subsequent variational problem tractable remains to be seen.

One of us, A.M.S., wishes to thank Professor Fritz Busse for drawing his attention to the problem considered here during his year visit (1 September 1977 to 31 August 1978) to IGPP at UCLA. During that time he benefited from many stimulating discussions on bounding methods and the nature of his multi-α solutions. Noteworthy is the 18 year period before the emergence of a solution!

Appendix

The inner problem for each boundary layer except the Nth reduces to solving the dual regime problem

$$\hat{\Omega}'''' - G\hat{\Theta} = 0, \tag{A 1}$$

$$\hat{\Theta}'' + G\hat{\Omega} = 0, \tag{A 2}$$

where $G = 1$ for $0 \leqslant \xi \leqslant \xi^*$ and $\hat{\Omega}\hat{\Theta} = 1$ (G unspecified) for $\xi^* \leqslant \xi \leqslant \infty$. The boundary conditions are that $\hat{\Omega} = \hat{\Omega}' = \hat{\Theta} = 0$ at $\xi = 0$, and $\hat{\Omega}' \to$ const. as $\xi \to \infty$, together with the matching conditions at $\xi = \xi^*$ that $\hat{\Theta}$, $\hat{\Theta}'$, $\hat{\Omega}$, $\hat{\Omega}'$, $\hat{\Omega}''$, and $\hat{\Omega}'''$ are continuous, $\hat{\Omega}\hat{\Theta} = 1$, and $(\hat{\Omega}\hat{\Theta})' = 0$. From (A 1) and (A 2), we find

$$\int_0^\infty G\,d\xi - \int_0^\infty G(1-\hat{\Omega}\hat{\Theta})\,d\xi = \int_0^\infty \hat{\Omega}''^2 d\xi = \int_0^\infty \hat{\Theta}'^2 d\xi = \beta. \tag{A 3}$$

A modification of Howard's (1963) argument leading to his (62) gives

$$(\hat{\Omega}''^2 - 2\hat{\Omega}'\hat{\Omega}''' + \hat{\Theta}'^2)' = 2\hat{\Theta}'\hat{\Theta}'' - 2\hat{\Omega}'\hat{\Omega}'''' = -2G(\hat{\Omega}\hat{\Theta})' = \begin{cases} -2(\hat{\Omega}\hat{\Theta})', & 0 \leqslant \xi \leqslant \xi^* \\ 0, & \xi^* \leqslant \xi \leqslant \infty. \end{cases} \tag{A 4}$$

Integrating the latter equation twice and using (A 3) leads to

$$\tfrac{1}{3}\int_0^\infty G\,d\xi = \tfrac{1}{2}\int_0^\infty (1-\hat{\Omega}\hat{\Theta})\,d\xi = \beta. \tag{A 5}$$

REFERENCES

BUSSE, F. H. 1969*a* On Howard's upper bound for heat transport by turbulent convection. *J. Fluid Mech.* **37**, 457–477.

BUSSE, F. H. 1969*b* Bounds on the transport of mass and momentum by turbulent flow between parallel plates. *Z. Angew. Math. Phys.* **20**, 1–14 (referred to herein as B69).

BUSSE, F. H. 1970 Bounds for turbulent shear flow. *J. Fluid Mech.* **41**, 219–240 (referred to herein as B70).

BUSSE, F. H. 1978 The optimum theory of turbulence. *Adv. Appl. Mech.* **18**, 77–121.

Howard, L. N. 1963 Heat transport by turbulent convection. *J. Fluid Mech.* **17**, 405–432.

Howard, L. N. 1972 Bounds on flow quantities. *Ann. Rev. Fluid Mech.* **4**, 473–494.

Ierley, G. R. & Malkus, W. V. R. 1988 Stability bounds on turbulent Poiseuille flow. *J. Fluid Mech.* **187**, 435–449.

Kim, H. T., Kline, S. J. & Reynolds, W. C. 1971 The production of turbulence near a smooth wall in a turbulent boundary layer. *J. Fluid Mech.* **50**, 133–160.

Malkus, W. V. R. 1954 The heat transport and spectrum of thermal turbulence. *Proc. R. Soc. Lond.* A **225**, 185–195.

Malkus, W. V. R. & Smith, L. M. 1989 Upper bounds on functions of the dissipation rate in turbulent shear flow. *J. Fluid Mech.* **208**, 479–507.

Smith, L. M. 1991 Turbulent Couette flow profiles that maximise the efficiency function. *J. Fluid Mech.* **227**, 509–525.

Townsend, A. A. 1956 *The Structure of Turbulent Shear Flow.* Cambridge University Press.